SISO Controller of a Bicycle-Rider System

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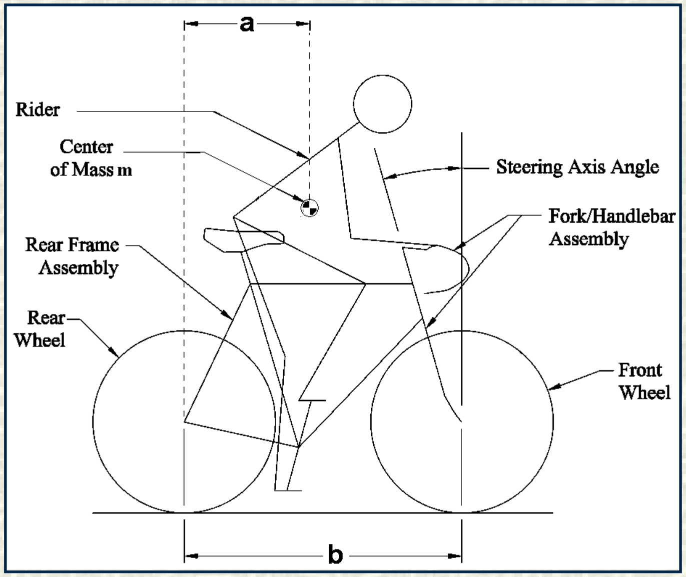
Objectives

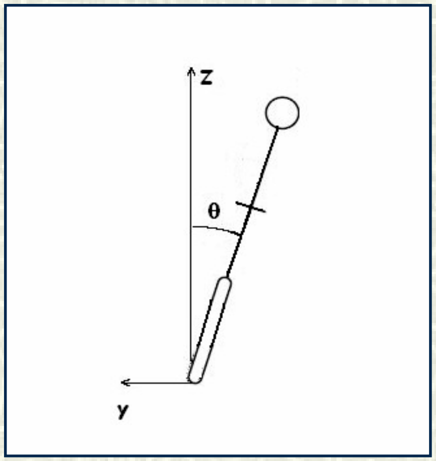
- To design a control system to keep a bicycle-rider system upright by varying only the steering angle.
- To compare three different control techniques:
  - Pole placement
  - LQR
  - Classical control

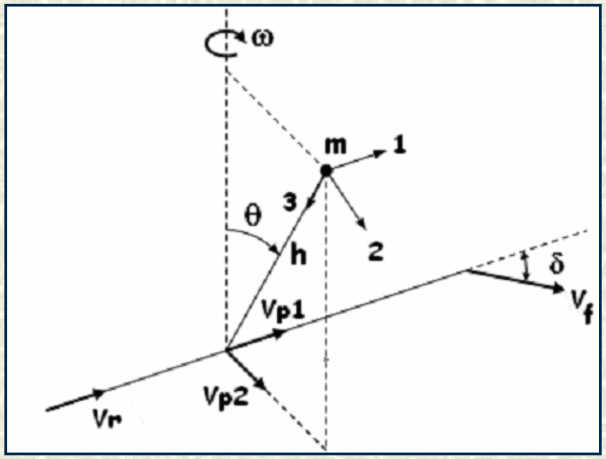
Model Description



- Inverted pendulum bicycle model
- One degree of freedom – roll angle (θ)
- EOM derived with Lagrange's method
- 2nd order system

- θ = roll angle
- δ = steering angle





Assumptions and Linearization

- One rigid body representing rider and bicycle
- Negligible mass ideal rolling wheels with no sideslip
- Constant rear wheel velocity
- Inherent stability characteristics due to frame geometry are neglected
- Rolling on a flat level ground plane
- Both the roll and steering angles are assumed to be small

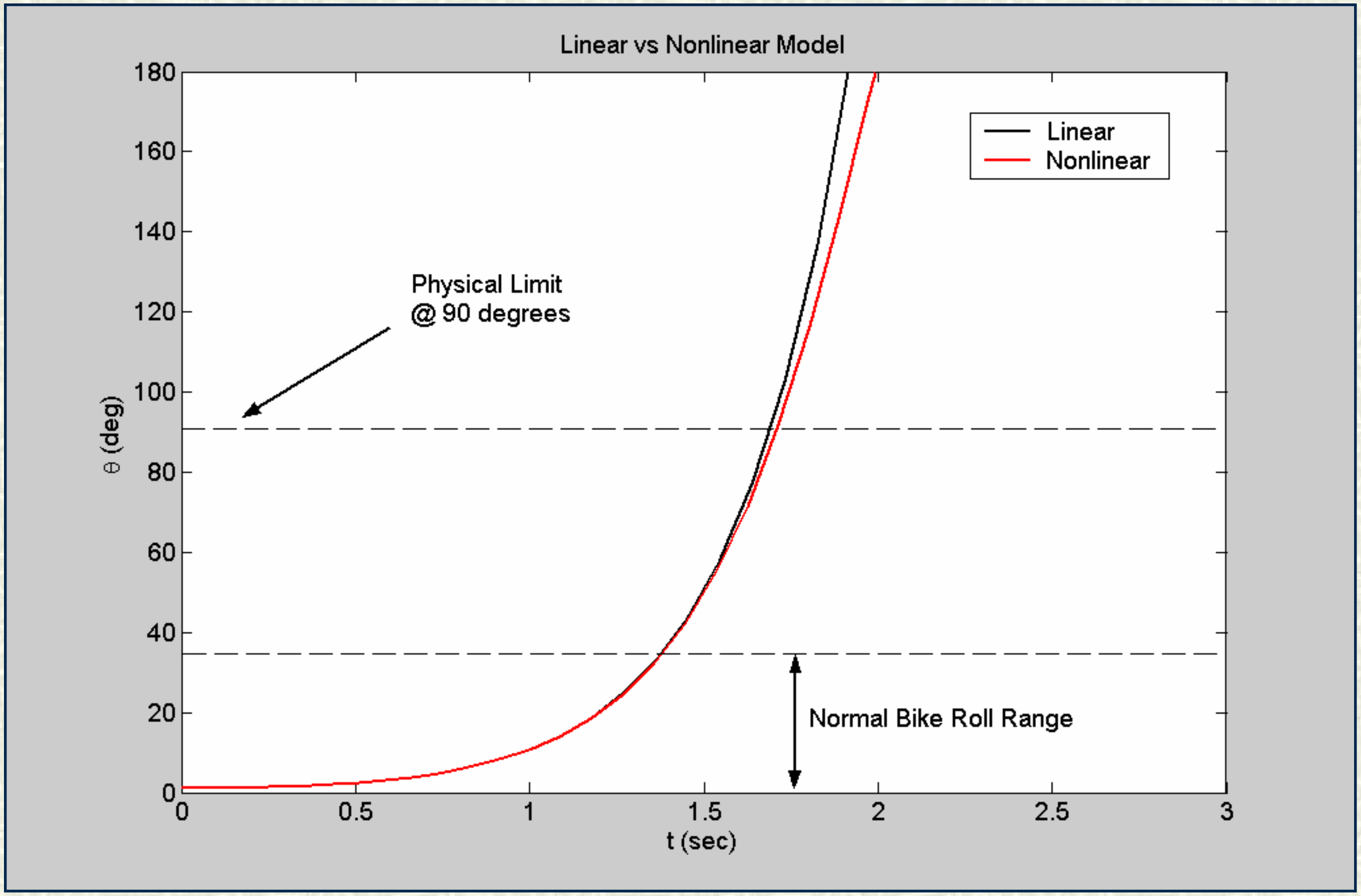
2nd Order Equation of Motion:

$$\left(I_1 + mh^2\right)\ddot{\theta} + \left(I_3 - I_2 - mh^2\right)\left(\frac{v_r \tan\delta}{b}\right)^2 \sin\theta\cos\theta - mgh\sin\theta = -mh\cos\theta\left(\frac{av_r}{b\cos^2\delta}\dot{\delta} + \frac{v_r^2}{b}\tan\delta\right)$$

⬇

$$\left(I_1 + mh^2\right)\ddot{\theta} - mgh\theta = -\frac{mh}{b}\left(av_r\dot{\delta} + v_r^2\delta\right)$$

Non-linear vs. Linear Model



State Space

- A derivative of the input is present in EOMs:

$$\left(I_1 + mh^2\right)\ddot{\theta} - mgh\,\theta = -\frac{mh}{b}\left(av_r\dot{\delta} + v_r^2\delta\right)$$

- A hand-derived *Controller Companion* form yields:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{mgh}{I_1 + mh^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\delta$$

$$\theta = \begin{bmatrix} \dfrac{-mhav_r}{b\left(I_1 + mh^2\right)} & \dfrac{-mhv_r^2}{b\left(I_1 + mh^2\right)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\delta$$

- But now the physical significance of the states is lost.

Observer Companion Form

- Converting to this form yields:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgh}{I_1 + mh^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{-mhav_r}{b\left(I_1 + mh^2\right)} \\ \dfrac{-mhv_r^2}{b\left(I_1 + mh^2\right)} \end{bmatrix} \delta$$

$$\theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\delta$$

- Now, the output θ **IS** the state x_1
- Specifying $x_1(0)$ is specifying the initial bike roll angle, $\theta(0)$.

Stability, Controllability, & Observability

- Unstable system: $$s_{1,2} = \pm\sqrt{\frac{mgh}{I_1 + mh^2}}$$

- Controllability: $$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -2.4 & -24.1 \\ -24.1 & -22.8 \end{bmatrix}$$

  rank(P) = 2

- Observability: $$N = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Control Design Criteria

- From an initial roll angle of 5° and a constant velocity of 5m/s, meet the following performance criteria:

  Overshoot < 1 degree

  Settling time < 2 sec

  Steer Input < 20 degrees

Full State Feedback & Regulator Design

- Letting the desired state vector be zero:

$$x_{error} = x_d - x = -x$$

- Now the control law reduces to

$$u(t) = -Kx(t)$$

- The close-loop state equations become

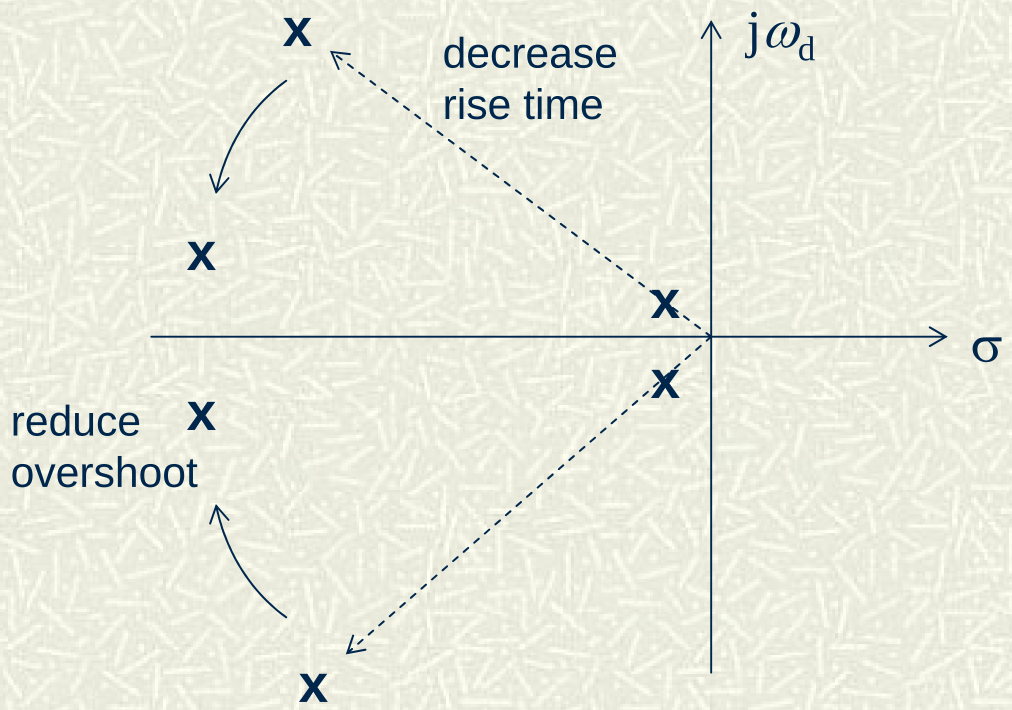
$$\dot{x}(t) = (A - BK)x(t)$$

- Consider design of the regulator by
  1. pole placement
  2. optimal control

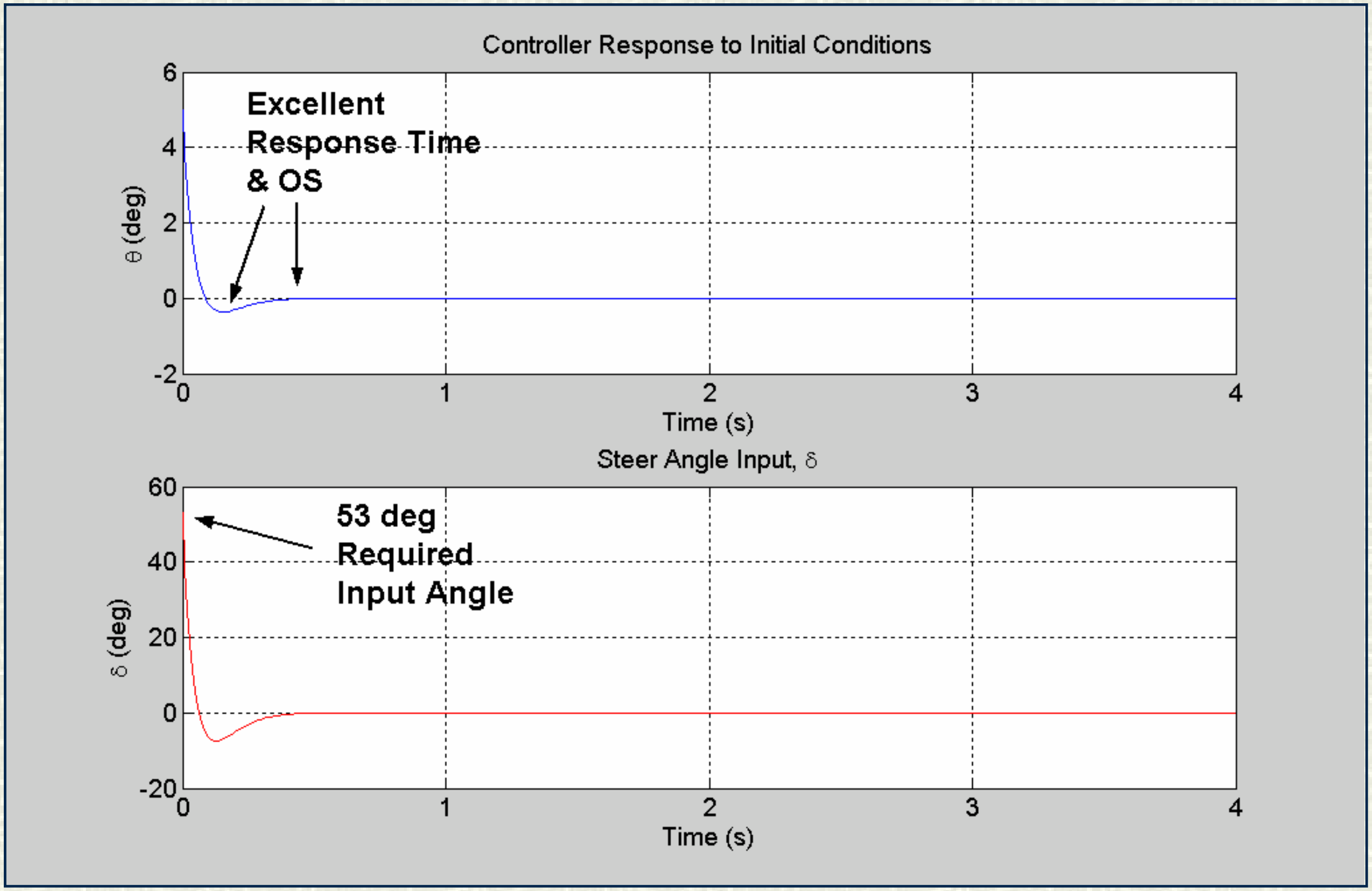
Pole-Placement

- Performance requirements trade-offs:

Speed of Response vs. Overshoot



Performance vs. Cost of Control



Butterworth Pattern

- Place poles on a radius R centered @ origin.
- Poles obtained from the solution of:

$$(s/R)^{2n} = (-1)^{n+1}$$

where *n* is the order of the system

- For *n*=2 the poles are the solutions of:

$$(s/R)^2 + (s/R)\sqrt{2} + 1 = 0$$

Final Pole Placement Design

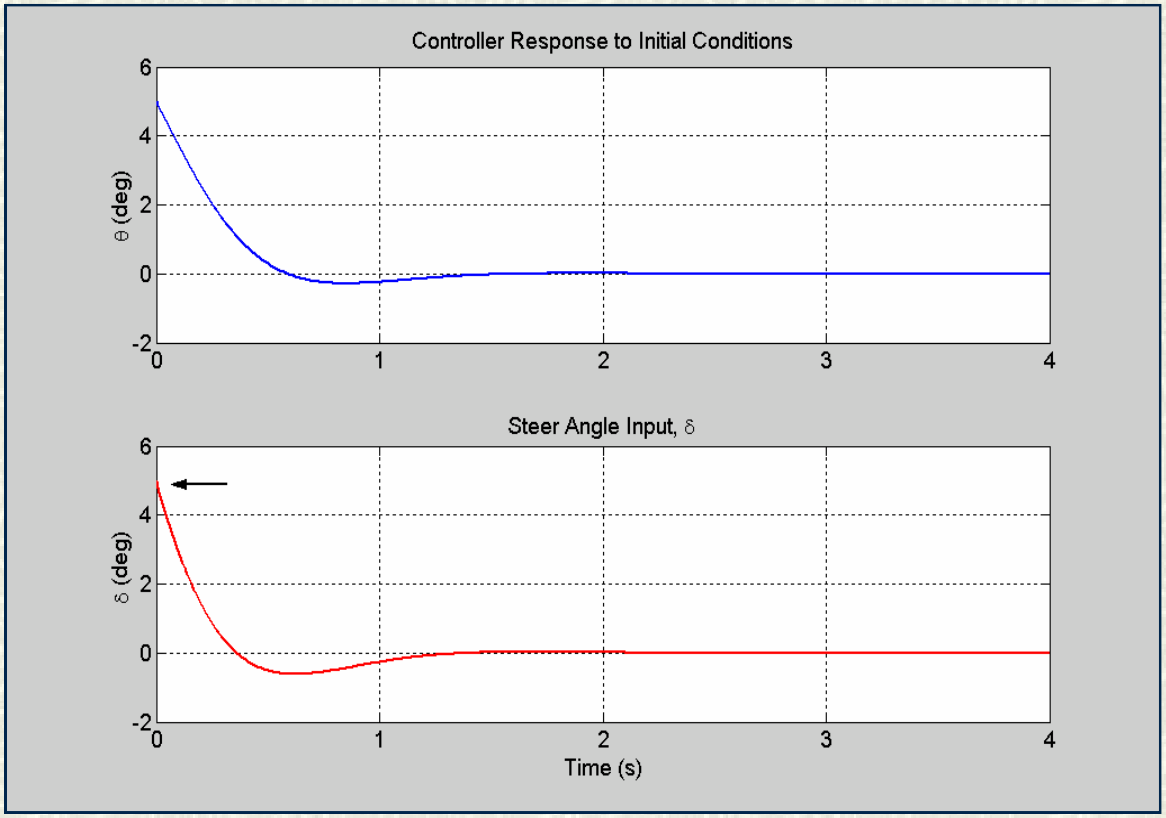
$$s_{1,2} = -3.01 \pm 3.01i \quad \Rightarrow \quad K = \begin{bmatrix} -1.0009 & -.1494 \end{bmatrix}$$

$$OS = .28\deg$$

$$t_p = .8\sec$$

$$T_s = 1.5\sec$$

$$\delta_{\max} = 5\deg$$



LQR Control

- Physical meaning of states
- Physical limits on δ: ±20º
- Physical limits on θ: ±35º
- When x_1 and δ were weighted equally and x_2 was weighted less the controller provided good results

$$J(u) = \int_0^T \left(\vec{x}^T R \vec{x} + \vec{u}^T \Lambda \vec{u} \right) dt$$

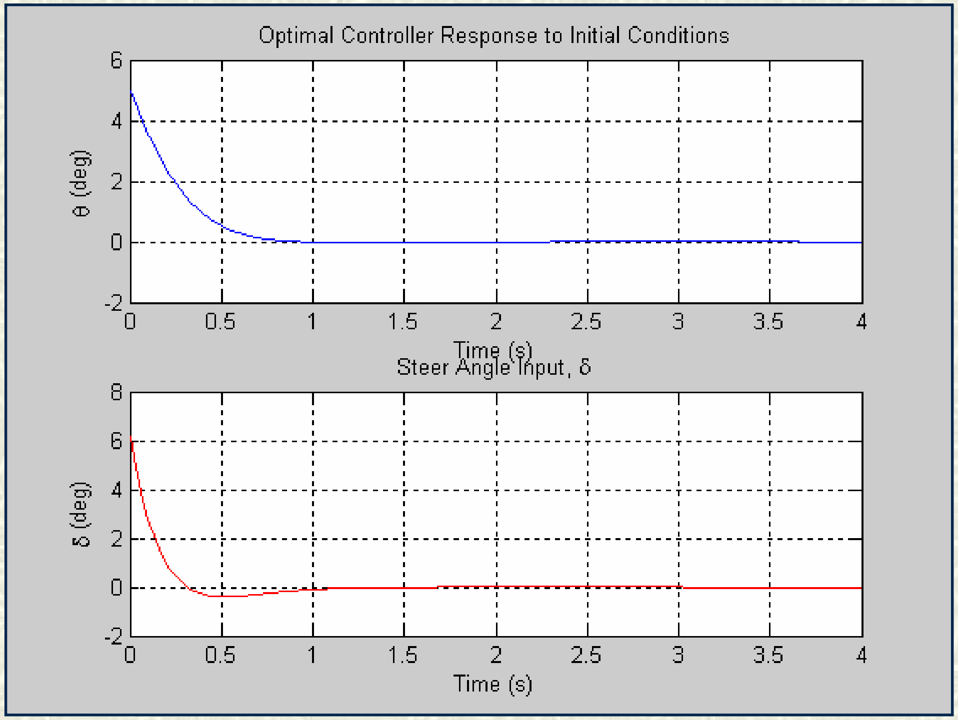
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} + \frac{mhav_r}{b\left(I_1 + mh^2\right)}\delta \end{aligned}$$

$$\vec{u} = [\delta]$$

Underdamped Case

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} \quad \Lambda = 1 \quad \Rightarrow \quad K = \begin{bmatrix} -1.2308 & -.2539 \end{bmatrix}$$

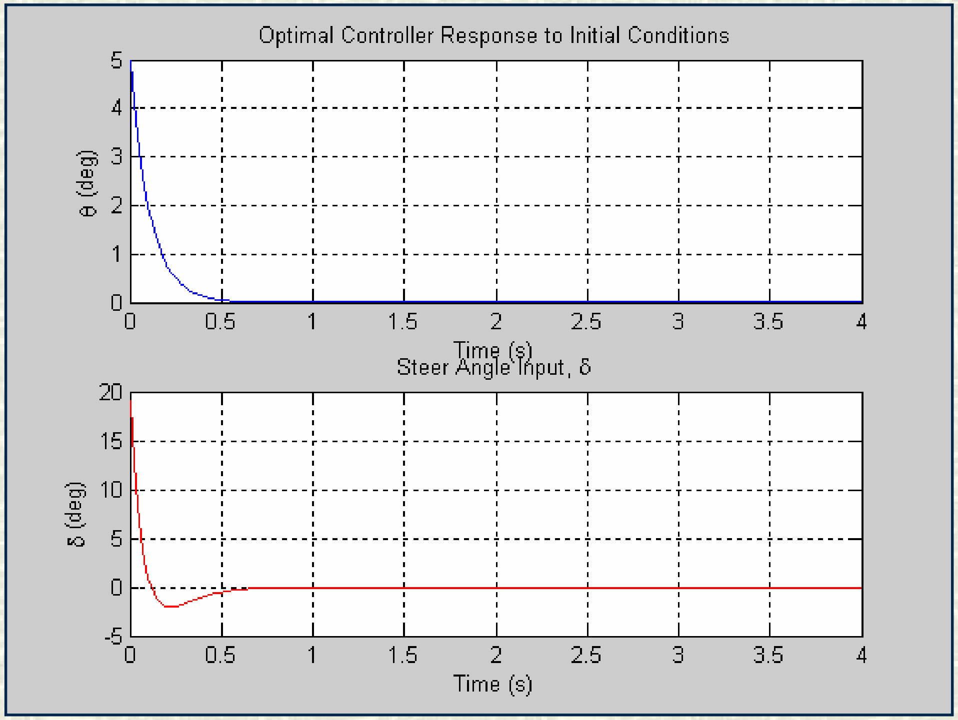
- $s_{1,2}$ = -4.54±2.32i
- OS = -0.02 deg
- t_p = 1.1 sec
- T_s = 0.86 sec
- δ_{max} = 6.2 deg



Overdamped Case

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} \quad \Lambda = 0.07 \quad \Rightarrow \quad K = \begin{bmatrix} -3.8165 & -.4153 \end{bmatrix}$$

- s_1= -10.05
- s_1 = -9.14
- T_s = 0.60 sec
- δ_{max} = 19.1 deg



Classical Control Analysis

- Linear model (SISO system)

$$\left(I_1 + mh^2\right)\ddot{\theta} - mgh\theta = -\frac{mh}{b}\left(av_r\dot{\delta} + v_r^2\delta\right)$$

- Laplace transform of linear model equation.

$$(I_1 - mh^2)s^2\theta(s) = -\frac{mhv_r}{b}(as + v_r)\delta(s)$$

- Transfer function

$$\frac{\theta}{\delta} = \frac{-\frac{mhv_r}{b}(as + v_r)}{(I_1 + mh^2)s^2 - mgh}$$

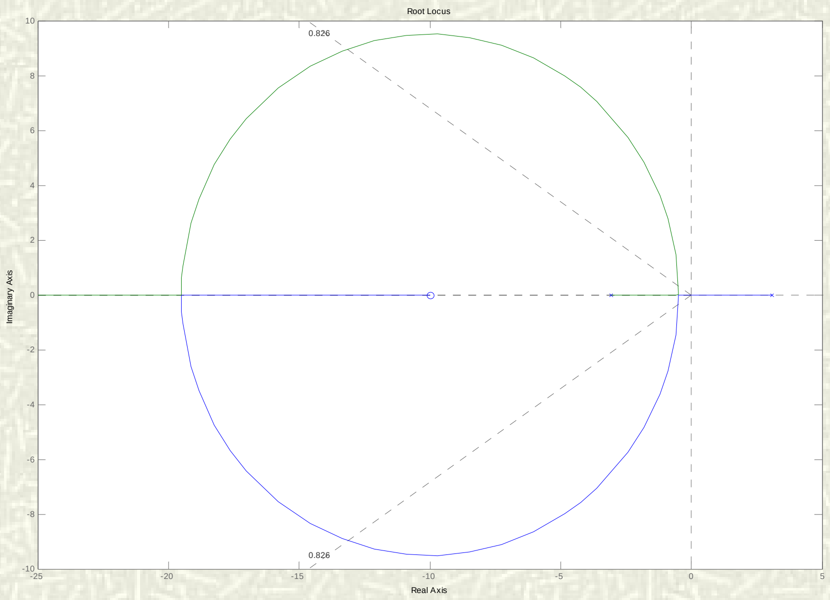
- As shown the system has a pole in RHP
- The system is unstable in roll angle without some control of steering angle.

Classical Control Analysis

For proportional control Routh- Hurwitz criteria to determine range of stability for K

S^2	1	9.42-K348
S^1	-87K	0
S^0	9.42-K348	0

$$\therefore K < 0$$



Root Locus of system proportionally controlled

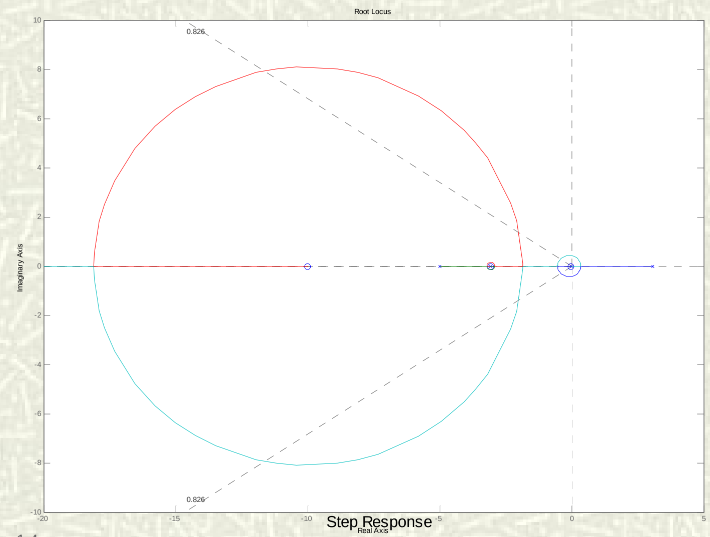
Classical Control Analysis

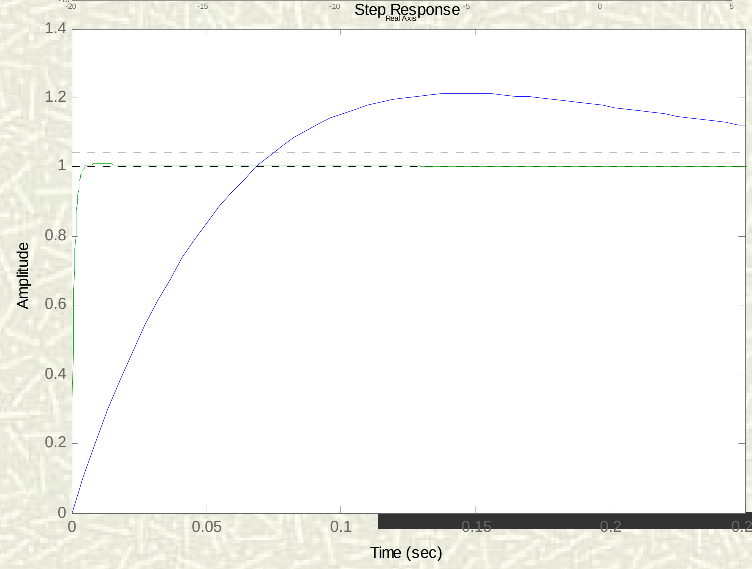
- Transient response improvement required: Lead controller design
- Decrease of steady state error : Lag controller design
- Controller TF

$$Gc = \frac{39.65\left(\left(S+3.07\right)(S+0.05)\right)}{(S+5)(S+0.01)}$$

- Open loop transfer function

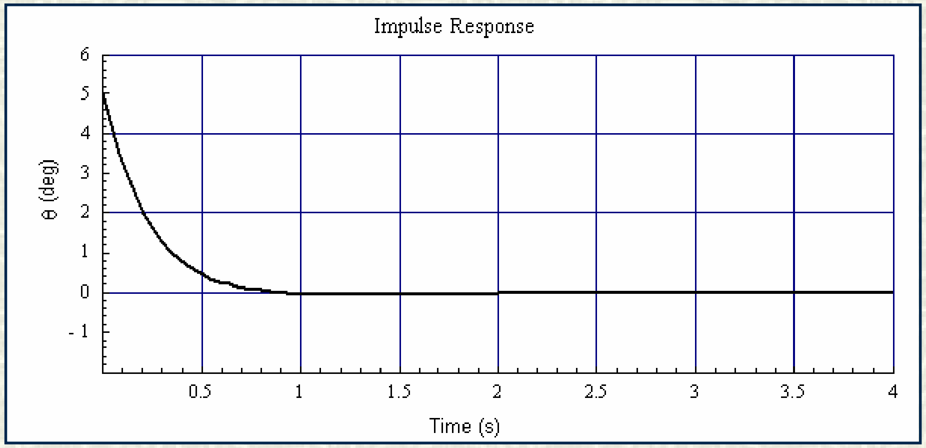
$$\left(Gc\right)\left(G\right) = \frac{39.65\left(\left(S+3.07\right)(S+10)(S+0.05)\right)}{(S+5)(S+3.07)(S-3.07)(s+0.01)}$$

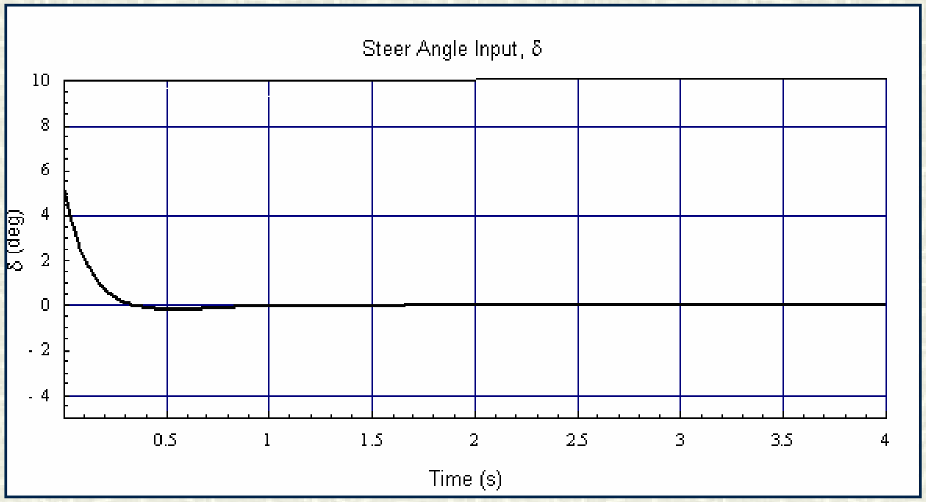




Classical Control Analysis

- System response to a initial lean angle of 5 deg.
- Ts=0.75 sec
- OS=-0.05 deg
- δmax = 5.1 deg
- Dominant poles (CLTF)
- S=-3.83+1.71j
- S=-3.83-1.71j





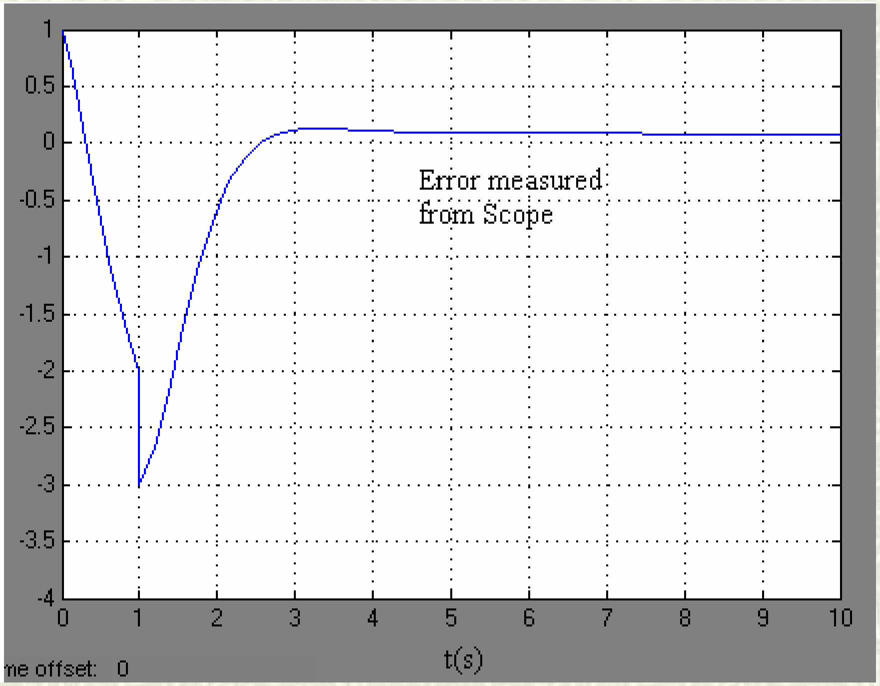
Conclusions

- Full-State Feedback is overkill for this SISO system
- Classical methods provide more tools for design in SISO than using MIMO methods
- LQR control methods are difficult to apply to when no physical limitations are in place

Questions???

Bicycle Parameters

- $I = 3.28$ (kg•m^2)
- $m = 87$ (kg)
- $h = 1$ (m)
- $a = 0.5$ (m)
- $b = 1.0$ (m)
- $v_r = 5$ (m/s)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